Surfaces

plane — linear surface

Spheres

Sphere Equation

$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2$] Cartesian

Vector eq:

$\vec{x} = (x, y, z) \quad \vec{x_0} = (x_0, y_0, z_0)$

$r^2 = \|\vec{x} - \vec{x}_0\|^2 = (\vec{x} - \vec{x}_0)\cdot(\vec{x} - \vec{x}_0)$

= sq. of dist from $\vec{x}$ to $\vec{x_0}$

$= \vec{x}\cdot\vec{x} + \vec{x}_0\cdot\vec{x}_0 - 2\vec{x}\cdot\vec{x}_0$

$x^2 + y^2 + z^2 + ax + by + cz + d = \emptyset$

Q Given such an eq, does it define a sphere?

A Not necessarily

e.g. $x^2 + y^2 + z^2 + 1 = \emptyset$

$\iff x^2 + y^2 + z^2 = -1$

$\Rightarrow$ empty set in $\mathbb{R}^3$

e.g. $x^2 + y^2 + z^2 = \emptyset$

$\Rightarrow$ point (ie sphere of radius $\emptyset$)

- Given $x^2 + y^2 + z^2 + ax + by + cz + d$, complete the square to figure out what it is

Intersections

sphere and a line: get 0, 1, or 2 pts

Algebraically, easiest way to solve is to write in parametric form

$\vec{x} = (x, y, z) = \vec{x_1} + t\vec{v}$

then plug this into the eq for sphere to get a quadratic eq from it

$\vec{x}_1$ = pt on line, $\vec{x}_0$ = center of sphere

→ Eq for t

$\vec{x} = \vec{x}_1 + t\vec{v}$

→ Eq for t

$\vec{x} = \vec{x}_1 + t\vec{v}$

$r^2 = (\vec{x} - \vec{x}_0) \cdot (\vec{x} - \vec{x}_0)$

$= (t\vec{v} + \vec{x}_1 - \vec{x}_0) \cdot (t\vec{v} + \vec{x}_1 - \vec{x}_0)$

$= (\vec{v} \cdot \vec{v})t^2 + 2(2(\vec{x}_1 - x_0) \cdot \vec{v})t + (\vec{x}_1 - \vec{x}_0) \cdot (\vec{x}_1 - \vec{x}_0)$

Sphere and a Sphere

Intersection is either:

1. a circle
2. a point
3. empty

Easier way to find intersection

$x^2 + y^2 + z^2 + a_1x + b_1y + c_1z + d_1 = \emptyset$

$x^2 + y^2 + z^2 + a_2x + b_2y + c_2z + d_2 = \emptyset$

→ subtract top from bottom

$(a_1 - a_2)x + (b_1 - b_2)y + (c_1 - c_2)z + d_1 - d_2 = 0$

⇒ eq of a plane

⇒ intersection of the spheres is the intersection of one sphere with that plane

→ solve for one of x, y, or z then plug into the other eq

eg. if $a_1 - a_2 \neq 0$, can solve for x

but if $a_1 - a_2 = 0$, solve for y

Q what if $a_1 - a_2 = b_1 - b_2 = c_1 - c_2 = \emptyset$?

A this happens if the two spheres are concentric

then, either:

1. radii are different
   → empty intersection
2. radii are same
   → they are the same sphere
   → intersection in a sphere

Cylinder

e.g. $(x-a)^2 + (y-b)^2 = r^2$

eg $(y-b)^2 + (z-c)^2 = r^2$ another cylinder

Intersections

Cylinder ; xy Plane

| its intersection w/ xy-plane is a

its intersection w/ xy-plane is a circle of radius r.

↳ <u>Def</u>

the intersection of a surface w/a plane is a trace of that surface

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Quadric Surfaces

↳ <u>Def</u>

Anything given by an eqn of the form

$$ax^2 + by^2 + cz + dxy + eyz + Fxz + gx + hy + iz + j = \varnothing$$

↳ <u>Examples</u>

- sphere & cylinder
- <u>ellipsoid:</u>

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

<u>equivalently:</u>

$$\alpha x^2 + \beta y^2 + \gamma z^2 = \delta \qquad \delta \neq \varnothing$$

can be put into the form above

Traces are ellipses

- <u>hyperboloid</u>

① one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

② two sheets

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

<u>Traces</u> of hyperboloids are conic sections

ie — ellipses, hyperbola & parabola

<u>Q</u> How to find?

On a plane parallel to coord plane just set x, y, or z to be a constant and then get eq of trace in the other var just set x, y, or z to be a constant and then get eq of trace in the other var.

eg One-sheet hyperboloid

→ if we set $z = \text{const}$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{z^2}{c^2}$$

$\Rightarrow$ ellipse

→ if we set $y = \text{const}$

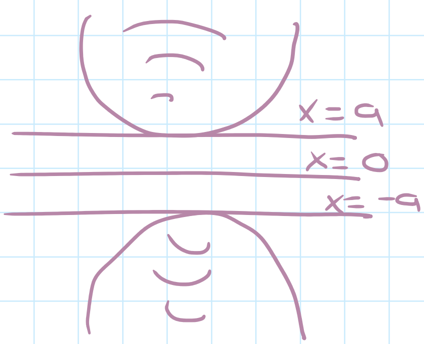
$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{y^2}{b^2}$$

eg two-sheet

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

$$\Rightarrow \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{x^2}{a^2} - 1$$

→ if set $x = \text{const}$ (ie, plane ∥ to yz plane) then either



btwn the sheets → ① $x^2 - a^2 < \varnothing$

→ trace is empty

tangent to one sheet ② $x^2 - a^2 = 0$

→ trace is a point

③ $x^2 - a^2 > 0$

→ trace is an ellipse

see also: elliptic paraboloid.
hyperbolic paraboloid

- elliptic cone:
  like 2 cones, one of them upside down meeting at a point

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = \varnothing$$

Remark

$x^2 \quad y^2 \quad z^2 = d$

Remark

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = d$$

defines
1. hyperboloid of one sheet if $d>0$
2. hyperboloid of two sheets if $d<0$
3. elliptic cone if $d=0$

Ruled Surfaces

↳ def
A surface is ruled for any point P on the surface, there's

e.g. a cylinder is ruled

Given any (x_0, y_0, z_0) on the cylinder
$(x-a)^2 + (y-b)^2 = r^2$
the line given by the two eqns
$x = x_0$, $y = y_0$
and is contained in the cylinder

BUT sphere is not ruled.
Q. Why?
A. bc no line is contained wholly in the sphere

Doubly ruled

↳ given any point, there are two distinct lines through that point contained in the surface

"regulus"

Curvilinear Coordinates

Cylindrical coords

Cylindrical Coords

↳ defined by

(r, θ, z) such that $\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = z \end{cases}$ $\quad r^2 = x^2 + y^2$

$\theta = \arcsin(y/r)$

→ like polar coord in x, y & don't do anything to z

Note $r \geq 0$
$0 \leq \theta \leq 2\pi$

Q: Why "cylindrical"?
A: b/c an eqn $r = \text{const}$ defines a cylinder

Cool geometric surface:
$z = \theta$ defines a "helicoid"
→ looks like parking garage

Spherical Coords (ρ, θ, ϕ) → (rho, theta, phi)

↖ same as cylindrical

$\rho^2 = x^2 + y^2 + z^2$

$\rho = \|(x, y, z)\|$

$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\cos\theta$
$z = \rho\cos\phi$

ϕ = angle from z-axis

↳ so $\phi = 0$ if on positive z-axis
$\phi = \pi$ if on negative z-axis
$\phi = \pi/2$ if on xy-plane

$$\boxed{\begin{array}{l} 0 \leq \theta < 2\pi \\ 0 \leq \phi \leq \pi \end{array}}$$

↳ so $\boxed{r = \rho\sin\phi}$ relation btwn spherical & cylindrical coords

Q: Why "spherical"?
A: Bc eq $\rho = \text{const}$ defines a sphere centered at origin.

centered at origin.

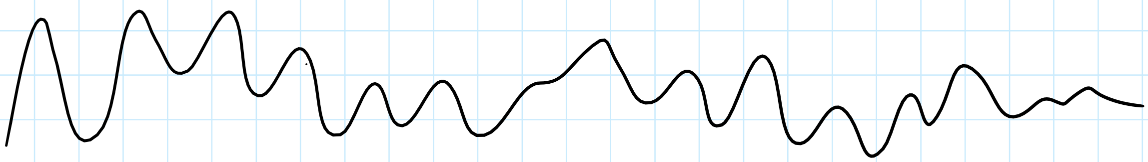
Q: What about sphere centered somewhere else?

A: See Example 1.33
(The eqn is REALLY messy)

Helix

$z = \theta \quad r = \text{const}$

$\Rightarrow$ a curve



Problems

Sec 1.2 C6

Q Can every vector in $\mathbb{R}^3$ be written as a linear combo of $\vec{i}$ and $\vec{j}$?
i.e. $\vec{v} = m\vec{i} + n\vec{j}$?

A no

eg $(0,0,1) = \vec{k}$ cannot be written this way

Why?
If $\vec{v} = m\vec{i} + n\vec{j}$ then the z coord of $\vec{v}$ must be $\emptyset$

Therefore if $\vec{v}$ has nonzero z-coord, then $\vec{v}$ cannot be written in that form

Sec 1.3 A6

Q angle btw $\underbrace{(4,2,-1)}_{=\vec{v}}$ & $\underbrace{(8,4,-2)}_{=\vec{w}}$

notice $\vec{w} = 2\vec{v}$

$\hookrightarrow$ $\vec{w}$ & $\vec{v}$ point in same dir

$\rightarrow$ angle btw them $= \emptyset$